J. Fluid Mech. (1968), *vol.* 31, *part* 4, *pp.* 753–778
Printed in Great Britain

The prediction of turbulent boundary layer development in compressible flow

By J. E. GREEN
Cambridge University Engineering Laboratory†

(Received 4 April 1967 and in revised form 4 August 1967)

Starting from Head's semi-empirical method for incompressible flow, two approaches to the prediction of turbulent boundary-layer development in compressible flow are explored. The first uses Head's incompressible method in conjunction with a compressibility transformation similar to Stewartson's transformation for laminar flow; the second carries over Head's physical arguments to treat the compressible flow directly. Measurements in supersonic flow, both on flat plates and downstream of an abrupt pressure rise, show broad agreement with the predictions of the second method but do not support the compressibility transformation. In particular, measurements on flat plates reveal that as Mach number increases the entrainment rate decreases to a lesser extent than the skin-friction coefficient. Whilst this result is consistent with the second treatment in this paper, it is difficult to reconcile with any of the compressibility transformations discussed, and the validity of these transformations in turbulent flow is therefore questioned.

1. Introduction

Methods for predicting the growth of a turbulent boundary layer in two-dimensional incompressible flow proliferate. However, because the structure of turbulent shear flow is not fully understood, all these methods are semi-empirical and none is entirely satisfactory. For the same reason, no really sound basis exists for adapting them to treat high-speed flows or flows with heat, or mass transfer at the surface.

Nevertheless, as flight speeds increase, the need for reliable predictions of turbulent boundary-layer phenomena in compressible flow grows. The present paper compares two attempts to meet this need by adaptations, based on different hypotheses, of a single incompressible calculation procedure. Perhaps the greatest value of this exercise is that it focuses attention on processes in the outer part of the boundary layer, revealing a trend in experimental data which might provide guidance for future work. At the same time, the more successful of the two procedures may have some practical worth as an interim prediction method.

The starting point is Head's (1960) method for incompressible flow, which empirically predicts the rate at which fluid is entrained from the free stream into

† Present address: Royal Aircraft Establishment, Bedford.

the boundary layer. In his extensive survey of calculation methods, Thompson (1964) found this treatment appreciably more successful in predicting experimentally observed flows than any alternative method available at the time. Moreover, although a number of procedures published since Thompson's work seem to be at least as successful as Head's, none has yet been shown to have a marked superiority.

The extension of this method to deal with compressible flow is approached in two contrasting ways. The first of these employs Mager's (1958) transformation, which is an adaptation to turbulent flow of Stewartson's (1949) compressibility transformation for the laminar boundary layer. The second extends the underlying assumptions of Head's method to embrace compressible flows by direct physical argument.

The predictions of the two methods are compared with some measurements, made by the author (Green 1966), of boundary-layer development in the constant-pressure region downstream of an abrupt, shock-induced pressure rise. Their application to supersonic flow over a flat plate is also considered, and their respective predictions of the variation of shape parameter with Mach number are compared with the experimental results of several workers.

In both cases experiment broadly supports the second calculation method in preference to that based on a compressibility transformation, and the trend of the flat plate data plainly conflicts with the implications of the transformation. The final section of the paper discusses this latter result in the context of transformations in general, with particular reference to the more sophisticated form first proposed by Coles (1962). It concludes that none of the transformations considered can be reconciled with the observed trend of flat plate shape parameter with Mach number.

2. Head's method for incompressible flows

For turbulent boundary layers in incompressible flow, Head (1960) derived a procedure for simultaneously calculating the development of the momentum thickness θ and a quantity herein referred to as mass flow thickness and written

$$\Delta = \int_0^\delta \frac{u}{u_e}\,dy = \delta - \delta^* \qquad (2.1)\dagger$$

where $\delta = y$ at $u/u_c = 0{\cdot}995$.‡

In this procedure, the momentum-integral equation

$$\frac{d\theta}{dx} = \frac{c_f}{2} - \frac{\theta}{u_e}\frac{du_e}{dx}(H+2) \qquad (2.2)$$

is integrated simultaneously with an auxiliary equation which accounts for the rate at which the boundary layer entrains fluid from the free stream

$$\frac{d\Delta}{dx} = F - \frac{\Delta}{u_e}\frac{du_e}{dx}. \qquad (2.3)$$

† The second equation is inexact, since it neglects the very small contribution to δ^* of the velocity defect in the region $y > \delta$.

‡ The definition of δ is discussed further in § 7.2.

Head argued that the non-dimensional entrainment parameter F was a unique function of the shape parameter $H_1 = \Delta/\theta$, and obtained the empirical relation shown in figure 1. He also provided a graphical relation between H_1 and the conventional shape parameter $H = \delta^*/\theta$, again obtained empirically. The

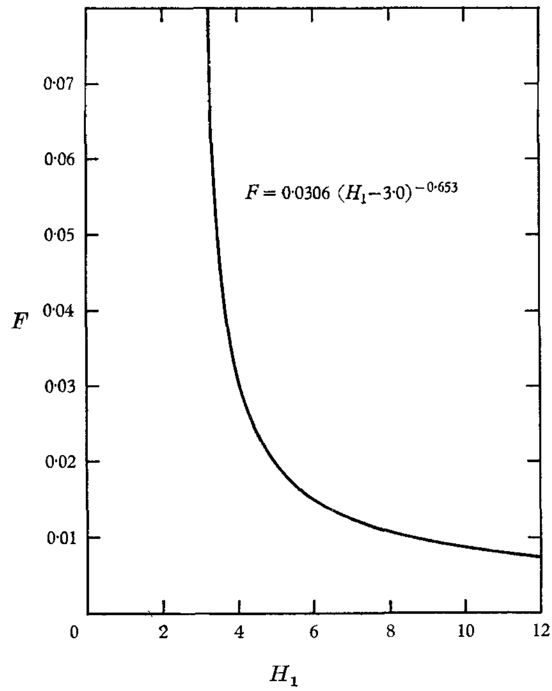


FIGURE 1. Head's (1960) empirical relation between entrainment and H_1.

choice of a suitable skin-friction relation, for example, that of Ludwieg & Tillmann (1949), completed the system of equations needed for a calculation.

In using this method with a transformation, the present author recast the auxiliary equation in a more conventional form.

Patel (1965) had already observed that the equation

$$H_1 = \frac{2H}{H-1} \tag{2.4}$$

which is true for velocity profiles of the one-parameter family $u/u_e = (y/\delta)^{1/n}$ fitted Head's graphical relation very well for values of H below about 2·2. The present author found that, by cross-plotting from Head's graphs, a linear relation was obtained between the conventional shape parameter H and the entrainment F,

$$F = 0{\cdot}025H - 0{\cdot}022, \tag{2.5}$$

which was again a good fit for H less than 2·2.

Then, by writing (2.3) as

$$H_1 \frac{d\theta}{dx} + \theta \frac{dH_1}{dx} = F - H_1 \frac{\theta}{u_e} \frac{du_e}{dx},$$

substituting for $d\theta/dx$ from the momentum-integral equation, and differentiating (2.4) to relate dH/dx and dH_1/dx, an auxiliary equation was obtained in the form

$$-\theta\frac{dH}{dx} = H(H^2-1)\frac{\theta}{u_e}\frac{du_e}{dx}+\frac{H-1}{2}[(H-1)F-Hc_f]. \tag{2.6}$$

This expression, together with the momentum-integral equation (2.2), an entrainment relation (2.5) and, say, the Ludwieg & Tillmann relation for the local skin-friction coefficient,

$$c_f = 0{\cdot}246\exp(-1{\cdot}561H)R_\theta^{-0{\cdot}268}, \tag{2.7}$$

provide a step-by-step method for calculating the development of an incompressible turbulent boundary layer.

To check that the analytic approximations to Head's curves did not significantly affect the method, a calculation by Thompson, using Head's graphs, of a boundary-layer development measured by von Doenhoff and Tetervin (figure 23 of Thompson's (1964) paper) was compared with a recalculation using (2.5) and (2.6). No significant difference was found.

3. Application of a compressibility transformation

3.1. *Mager's transformation*

Mager's (1958) compressibility transformation, as used here, (with $\gamma = 1{\cdot}4$) is defined by the relations

$$\left.\begin{aligned} d\bar{y} &= \left(\frac{T_e}{T_0}\right)^3\frac{\rho}{\rho_e}dy, & d\bar{x} &= \left(\frac{T_e}{T_0}\right)^4 dx,\\ \bar{u}/\bar{u}_e &= u/u_e, & \bar{u}_e &= M_e a_0, \end{aligned}\right\} \tag{3.1}$$

where unbarred symbols now refer to properties of a compressible flow, barred symbols denote quantities in the transformed plane and a is the speed of sound. The kinematic viscosity used in the evaluation of Reynolds number in the transformed flow is that of the real flow at stagnation temperature and pressure (suffix 0). As in Stewartson's treatment of the laminar boundary layer, the stream function is required to be invariant against the transformation, viscosity is taken as proportional to temperature, and the boundary-layer flow is assumed iso-energetic.

In his original paper, Mager (1958) presented more general relations than those above, claiming to improve the transformation by allowing a more accurate description of the variation of viscosity with temperature. However, following publication by Squire (1962) of an unfavourable comparison between skin-friction measurements in helium and the predictions of this more general form of the transformation, Mager (1962) reimposed Stewartson's restriction on viscosity variation. This reduced his transformation to the form of (3.1) and considerably improved its correlation of the helium skin-friction measurements.

3.2. *Boundary-layer calculation using the transformation*

The above transformation may be combined with the incompressible calculation method of the previous section to predict turbulent boundary-layer development in compressible flows. The procedure adopted by the author has been to transform the compressible problem (stated as a streamwise distribution of, for example, Mach number, given initial values of two independent boundary-layer parameters from which the variables $\bar{\theta}$ and $\bar{H}$ of equations (2.2) and (2.6) may be evaluated) into its equivalent incompressible one, predict boundary-layer development in the incompressible plane using equations (2.2), (2.5), (2.6) and (2.7), and transform the solution back to the compressible plane. All calculations have been performed on a digital computer, using an Adams–Bashforth library routine to integrate equations (2.2) and (2.6).

3.3. *Other transformations of Head's method*

Shortly after the above procedure had been developed, Standen (1964) published details of a similar treatment based on Head's method and the transformation of Culick & Hill (1958).

The difference between the two analyses lies chiefly in their respective transformations of the streamwise co-ordinate. In so far as Mager claims that his transformation is quite general and may be applied to flows with adverse pressure gradients, whereas Culick & Hill claim only to treat accelerating flows, there may be a slight case for preferring the present method to that of Standen. However, the difference between the two methods is small compared with the difference between either of them and the method described in the next section. It is for this reason that no comparisons with calculations by Standen's method are included in this paper.

Similarly, the more recent procedure of So (1965), which uses modified versions of Head's method and Coles's (1962) compressibility transformation, is not considered. In his comparisons with experiment, So found that his predictions showed 'no significant improvement' over those of Standen.

4. A calculation procedure based on a direct estimate of the entrainment in a compressible flow

4.1. *The basic hypothesis*

In his original paper on the turbulent boundary layer at low speeds, Head (1960) suggested that an extension of the method to compressible flow might be of interest. The treatment given in this section followed his reiteration of this suggestion to the present author.

The central question in making the extension is how Head's empirical relation between entrainment and the shape parameter H_1 should be treated. Head argued that in incompressible flow, 'the distribution of mean velocity in the boundary layer, in particular the velocity defect in its outer part (measured roughly by form parameters such as H and H_1), should control, or at least stand in close quantitative relationship with, the entrainment process'. Although he mentions the wake function of Coles (1956) as one (perhaps the most likely) of a number of approaches which might lead to a more logical prediction of entrainment, the recent paper of Escudier & Nicoll (1966) shows little benefit by adopting this approach. Assuming, then, that the original proposal was reasonably well founded, how should it be qualified in discussing compressible flows?

In his original paper Head tentatively suggested that the density variation in the outer part of a supersonic boundary layer would have little effect on the entrainment mechanism. We take up this suggestion by assuming that varying density does not change the relation which Head postulates between entrainment and the spatial distribution of velocity in the outer part of the layer. Further, we assume this velocity distribution is adequately specified by the shape parameter,

$$(H_1)_k = \int_0^\delta \frac{u}{u_e}\,dy \bigg/ \int_0^\delta \frac{u}{u_e}\left(1-\frac{u}{u_e}\right) dy \tag{4.1}$$

(where suffix k denotes a 'kinematic' property of the compressible layer). That is to say, entrainment F and the shape parameter $(H_1)_k$ are *assumed* to satisfy the empirical relation obtained by Head for incompressible flow, and F is now *defined* by the equation

$$F = \frac{1}{\rho_e\, u_e}\frac{d}{dx}\int_0^\delta \rho u\, dy. \tag{4.2}$$

There are three principal reasons for making this assumption here; it is simple, it is in line with Head's original suggestion, and it provides an alternative treatment to compare with that derived using the transformation. At the same time it is fairly sweeping and can properly be assessed only by comparison with experiment. There seems no good *a priori* reason to regard it as any more than a first-order attempt to treat the compressible problem.

There is, however, some evidence that as a first approximation it may not be seriously astray. Morkovin (1962) found strong evidence that the structure of the compressible layer is essentially the same as that found at low speeds. The transport mechanisms are dominated by the large eddies, with the large-scale velocity field driving the turbulent exchanges. Density variation appears to modulate rather than essentially change this structure, and to allow for this modulation some scaling of y is allowed when comparing compressible and incompressible flows. Thus compressibility transformations are not precluded. Nevertheless, when Morkovin compared measured distributions of non-dimensional shear stress and streamwise velocity fluctuations in high and low speed flows he found best agreement using the undistorted ordinate y/δ rather than any transformed scale. In itself this result has some affinity with the present entrainment assumption.

More recently Bradshaw & Ferriss (1966) have used Morkovin's arguments to extend their own 'turbulent energy' method to compressible flows. They concluded that in both high and low speed flows the entrainment rate is proportional to the maximum value of τ/ρ in the interval $y > \frac{1}{4}\delta$. For constant pressure flows this maximum occurs at the inner boundary $y = \frac{1}{4}\delta$. In another recent paper, Maise & McDonald (1967) have surveyed the experimental evidence and deduced that for flat plate boundary layers the distribution across the layer of non-dimensional mixing length,

$$\frac{l}{\delta} = \frac{1}{\delta\,\partial u/\partial y}\left(\frac{\tau}{\rho}\right)^{\frac{1}{2}},$$

is almost independent of Mach number. Hence for constant pressure flows we might argue

$$F \propto \frac{1}{u_e^2}\left(\frac{\tau}{\rho}\right)_{y=\frac{1}{4}\delta} \qquad \text{(Bradshaw \& Ferris)}$$

$$\propto \left(\frac{\delta}{u_e}\frac{\partial u}{\partial y}\right)^2_{y=\frac{1}{4}\delta} \qquad \text{(Maise \& McDonald)}$$

so that entrainment depends only on the spatial distribution of velocity. With the further assumption of Head's method, that velocity profiles belong to a one-parameter family, we conclude that $((\delta/u_e)\,\partial u/\partial y)_{y=\frac{1}{4}\delta}$ and (therefore) F are functions of $(H_1)_k$ only.

This is indirect experimental evidence that (in constant pressure flows at least) the present entrainment assumption allows adequately for compressibility effects. Exploratory use of the assumption as the basis of a prediction method for $\bar{H}$ therefore seems justified. Whether and in what circumstances it is a poor approximation will probably be discovered only by direct comparison with experiment.

4.2. *General relations*

From (4.2), the streamwise rate of increase of mass flow in a compressible boundary layer may be written

$$\frac{d}{dx}(\rho_e u_e \Delta) = \rho_e u_e F \tag{4.3}$$

where, in compressible flow,

$$\Delta = \int_0^\delta \frac{\rho u}{\rho_e u_e}\,dy = \delta - \delta^*.$$

Equation (4.3) may be rearranged

$$\frac{d\Delta}{dx} = F - \Delta\left(\frac{1}{\rho_e}\frac{d\rho_e}{dx} + \frac{1}{u_e}\frac{du_e}{dx}\right)$$

or, substituting

$$-\frac{M_e^2}{u_e}\frac{du_e}{dx} \quad \text{for} \quad \frac{1}{\rho_e}\frac{d\rho_e}{dx},$$

$$\frac{d\Delta}{dx} = F + (M_e^2 - 1)\frac{\Delta}{u_e}\frac{du_e}{dx} \tag{4.4}$$

and, with the same substitution, the momentum-integral equation for compressible flow is

$$\frac{d\theta}{dx} = \frac{c_f}{2} - (H + 2 - M_e^2)\frac{\theta}{u_e}\frac{du_e}{dx}. \tag{4.5}$$

It is worth noting that, provided F is the quantity defined by (4.2), equations (4.4) and (4.5) involve no assumptions other than the usual boundary-layer approximations.

4.3. *Auxiliary relations*

To allow solution of (4.4) and (4.5) given the streamwise distribution of Mach number, auxiliary relations are needed to provide local values of c_f, H and F.

In the present work, the immediate object was a comparison with some measurements made by the author (Green 1966) in a boundary layer which was recovering after interaction with an oblique shock wave. This comparison was intended primarily as a test of the assumption that Head's relation between F and H_1 could be used in these flows. Therefore the author's measurements were used wherever possible to provide *other* empirical relations needed in the analysis.

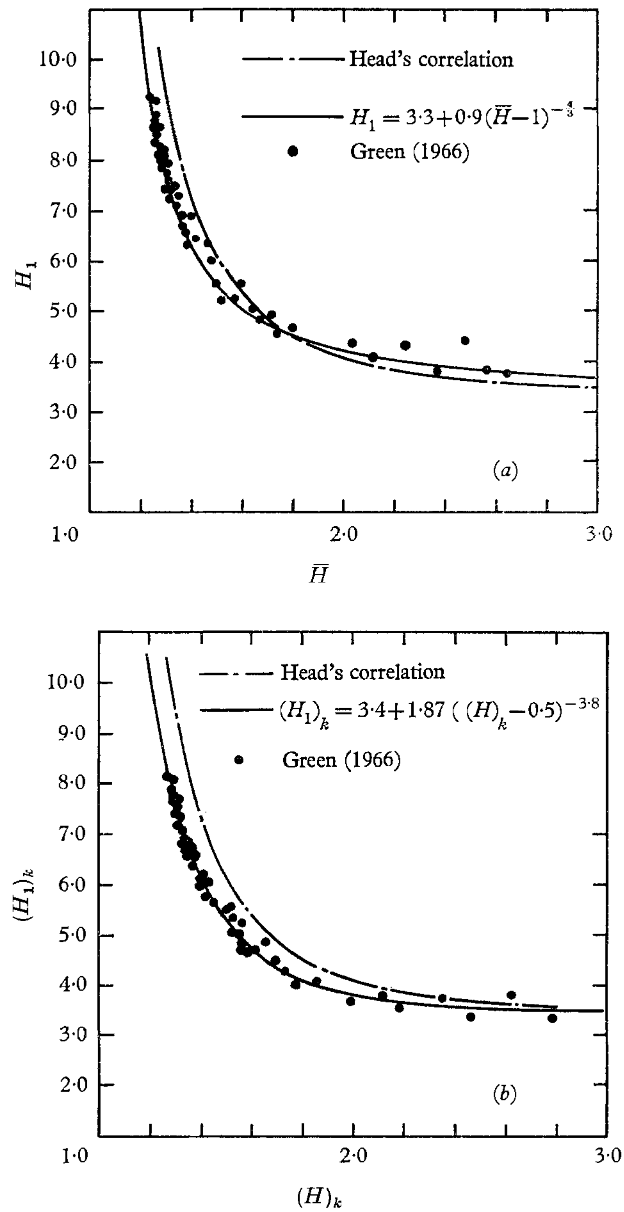


FIGURE 2. Correlations of transformed and kinematic shape parameters measured downstream of a shock wave/boundary-layer interaction.

Thus in figure 2*a* the values of the transformed shape parameter

$$\bar{H}\left(=\int_0^\infty \frac{\rho}{\rho_e}\left(1-\frac{u}{u_e}\right)dy\Big/\theta\right)$$

and of H_1 (the ratio Δ/θ of the two quantities calculated by the method) obtained from the author's measurements are plotted against each other. In figure 2*b* their kinematic equivalents $(H)_k$ and $(H_1)_k$ similarly obtained, are plotted.

We see that Head's incompressible correlation is not in very good agreement with either set of data, and that they are better fitted by

$$\bar{H} = 1 + (0{\cdot}9/(H_1 - 3{\cdot}3))^{0{\cdot}75} \tag{4.6}$$

and

$$(H_1)_k = 3{\cdot}4 + 1{\cdot}87/((H)_k - 0{\cdot}5)^{3{\cdot}8}. \tag{4.7}$$

In the appendix, relations are given between analogous parameters in the two above families, i.e. between integrals of the form

$$\int_0^\delta f\left(\frac{u}{u_e}\right)\frac{\rho}{\rho_e}\,dy$$

and their counterparts with density excluded,

$$\int_0^\delta f\left(\frac{u}{u_e}\right)dy.$$

These relations include one of the form

$$(H)_k = \bar{H}\phi(M_e, \bar{H}). \tag{4.8}$$

(See equation (A 5).)

Now the calculation procedure determines, as it advances, local values of Δ and θ. From their ratio H_1 we may determine $\bar{H}$ from (4.6), use the result of the appendix equation (A 5), to evaluate $(H)_k$, and then obtain $(H_1)_k$ from (4.7). Hence we may determine F, since we have assumed it to relate to $(H_1)_k$ in the manner proposed by Head. Standen (1964) has fitted Head's empirical curve with the equation

$$F = 0{\cdot}0306((H_1)_k - 3{\cdot}0)^{-0.653}. \tag{4.9}$$

To evaluate the remaining unknowns in the momentum integral equation, H and c_f, we follow Spence (1961). It is assumed that the temperature distribution through the boundary layer is given by the quadratic

$$T = T_w + (T_r - T_w)\frac{u}{u_e} + (T_e - T_r)\left(\frac{u}{u_e}\right)^2, \tag{4.10}$$

where suffices w and r indicate respectively wall and recovery temperatures.

This is probably a fair approximation in flows with appreciable pressure gradients but small heat transfer, or with small pressure gradients and isothermal walls. It would be expected to fail if there was a significant streamwise variation in wall temperature, or if heat transfer and pressure gradients were *both* appreciable. Its use makes the present treatment slightly more general than that of the previous section, which was restricted by the transformation to iso-energetic flows.

From this temperature distribution it follows that

$$H = \frac{T_w}{T_e}\bar{H} + \frac{T_r}{T_e} - 1 \tag{4.11}$$

where $\bar{H}$ is given by (4.6).

To determine c_f, Spence proposed the transformation of an incompressible skin-friction law to a reference state given by Eckert's (1955) intermediate temperature relation. Thus

$$c_f = \frac{T_e}{T_m}\bar{c}_f, \tag{4.12}$$

where T_m is the intermediate temperature

$$T_m = 0{\cdot}5\,(T_w + T_e) + 0{\cdot}22\,(T_r - T_e) \tag{4.13}$$

and $\bar{c}_f$ is a skin-friction coefficient evaluated from an incompressible law at a Reynolds number

$$\bar{R}_\theta = \frac{\mu_e}{\mu_m} R_\theta. \tag{4.14}$$

R_θ is the Reynolds number based on the actual momentum thickness and the free-stream properties of the compressible flow, and the viscosity ratio may be obtained from Sutherland's viscosity law. An appropriate incompressible skin-friction law is that of Ludwieg & Tillmann (1949). Consistent with Spence's arguments, the transformed shape parameter $\bar{H}$ is taken as the equivalent of the conventional shape parameter in incompressible flow. Thus

$$\bar{c}_f = 0{\cdot}246 \exp(-1{\cdot}561\,\bar{H})\,\bar{R}_\theta^{-0{\cdot}268}. \tag{4.15}$$

4.4. *Summary of the method*

Boundary-layer development is calculated by simultaneously advancing the entrainment equation, (4.4), and the momentum integral equation, (4.5).

At the end of each step new values of Δ and θ are obtained and local values of M_e and T_w are known. Then $H_1 = \Delta/\theta$, $\bar{H}$ is evaluated from (4.6), H_k from (4.8), $(H_1)_k$ from (4.7) and finally F from (4.9). Knowing $\bar{H}$, H is now determined from (4.11). T_m is evaluated from (4.15), μ_e/μ_m is determined from a suitable viscosity relation, and we evaluate, in turn, $\bar{R}_\theta$ (equation (4.14)), $\bar{c}_f$ (equation (4.15)) and c_f (equation (4.12)). The forward integration of (4.4) and (4.5) may now be taken a step further.

In using this method to predict boundary-layer development in the flows observed by the author, wall and stagnation temperature were taken as equal (since this was fortuitously the case in the experiments) and recovery temperature was obtained by assuming a recovery factor of 0·89 (a value widely used for turbulent flow). Computations were performed using the same library integration routine as for calculations under the transformation.

5. A comparison between measurements in a disturbed supersonic boundary layer and the predictions of the two methods

5.1. *The experimental results*

The experiments with which the two calculation methods are compared are to be reported fully elsewhere and will only be outlined here.

Briefly, a two-dimensional wedge of variable incidence was installed to span a supersonic wind tunnel and to produce a plane oblique shock wave which was reflected at the tunnel floor. The pressure on the floor rose steeply through the region of shock reflexion and thereafter remained nearly constant for an appreciable distance. Boundary-layer development through the pressure rise and the subsequent constant pressure region was observed by pitot traverse.

The Mach number of the undisturbed flow was 2·5 and the momentum thickness Reynolds number at the point of shock impingement was about 3×10^4. The flows produced by incident shocks with nominal deflexion angles of 2, $3\frac{1}{2}$, 5, $6\frac{1}{2}$, 8 and $9\frac{1}{2}$ degrees were studied.

For the weakest three shocks, the boundary layer remained attached throughout the interaction region. For shocks of $6\frac{1}{2}^\circ$ and above a closed separation bubble formed within the region of rising pressure, its streamwise extent increasing with shock strength. The measurements used here were all made in the attached flow downstream of the pressure rise, where the boundary layer was recovering towards its equilibrium form for zero pressure gradient.

Since the tunnel floor was plane and streamwise pressure gradients were small, $\partial p/\partial y$, the gradient normal to the floor, was also everywhere small and the use of a first-order boundary-layer analysis therefore justified. To test for cross-flows, the two-dimensional momentum integral equation was advanced (the integrand being evaluated from the experimental data) and the resulting distribution of momentum thickness compared with experiment. In almost all cases the disagreement was within the experimental scatter, and was small compared with changes in such other quantities as displacement thickness and shape parameter.

5.2. *Performance of the two methods*

The methods are compared in terms of their predictions of the transformed shape parameter $\bar{H}$. This parameter is an analogue, in compressible flow, of the conventional shape parameter H which is generally used as a test of calculation procedures in incompressible flow. Like the latter it is important in that it correlates strongly with skin-friction coefficient. Also, since its 'flat plate' value varies only slightly with Mach number, it provides a good qualitative indication of the local state of the boundary layer relative both to separation and to constant pressure equilibrium.

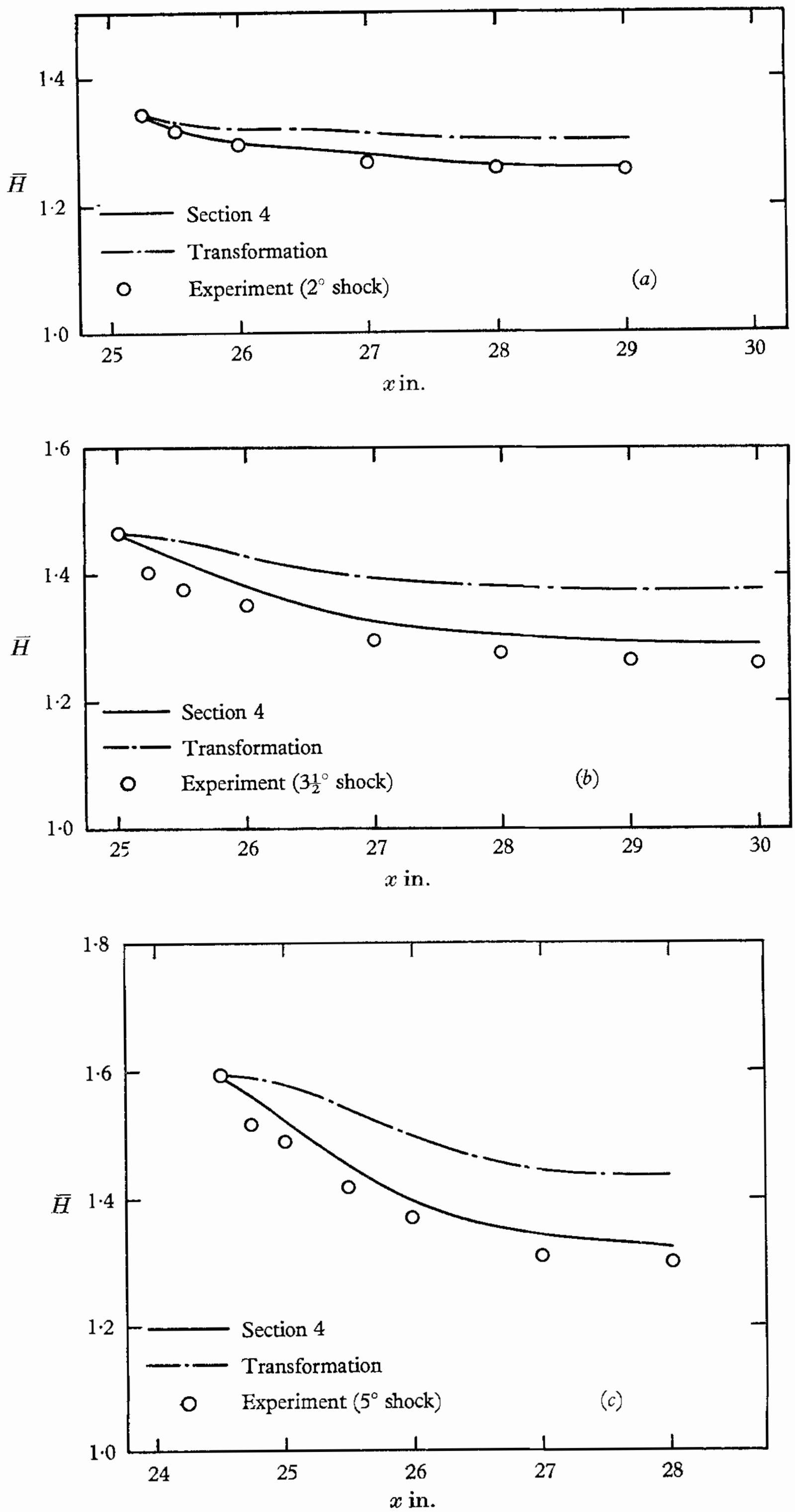
Figures $3a$ to $3f$ compare the calculated and observed distributions of $\bar{H}$. In every case there is a marked discrepancy between the two procedures.

The predictions of the method derived by extending Head's arguments to compressible flows are in general reasonably good. The initial rapid fall in $\bar{H}$ is fairly well described, even for the most severely disturbed flows, and it is only in the latter part of the calculation that the rate of recovery of $\bar{H}$ is underestimated. For the $9\frac{1}{2}^\circ$ shock (figure $3f$), the predicted value of $\bar{H}$ at the downstream limit of the measurements is about 10 % higher than the observed value. For the other shocks this discrepancy is less than 5 %.

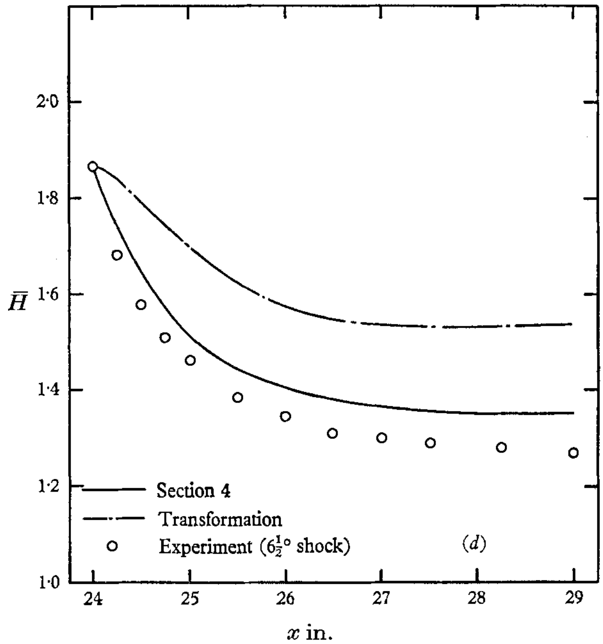
The predictions of the method using a transformation are indifferent by comparison. The predicted rate of decrease in $\bar{H}$ is appreciably too low and in some cases, where there is a slight adverse pressure gradient over the first part of the development, an initial increase is predicted. For the $9\frac{1}{2}^\circ$ shock, the predicted value of $\bar{H}$ at the most downstream measurement is about 40 % too high. For other shocks, the error at the downstream limit is generally more than four times the error of the other calculation.

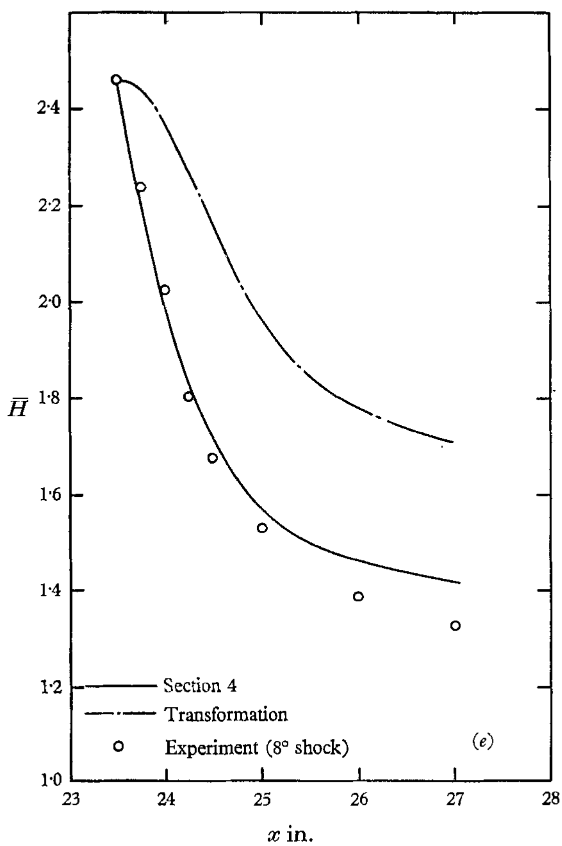
These results would seem at first sight to provide strong support for the direct extension of Head's method, and to cast doubt on the validity of the transformation.

However, Bradshaw & Ferriss (1965), in their study of an incompressible boundary layer which was recovering after a period of adverse pressure gradient,



(for legend see p. 766)





(for legend see p. 766)

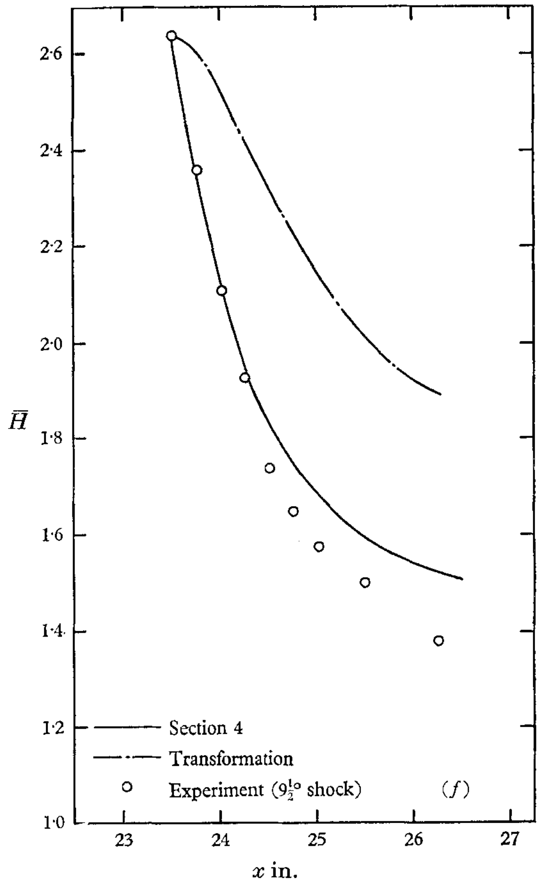


FIGURE 3. Comparison between the predictions of the two calculation methods and the observed shape parameter development downstream of a shock wave/boundary-layer interaction.

showed that Head's method underestimated the observed rate of recovery. We must therefore ask whether Head's method should be *expected* to predict the present flows satisfactorily.

In this context some qualitative differences between the experiments of Bradshaw & Ferriss (1965) and those of the present author (Green 1966) should not be overlooked. In Bradshaw's flow, the boundary layer developed in an 'equilibrium' pressure gradient for a considerable distance before entering the constant pressure region. Thus stress levels within the boundary layer had attained their equilibrium values in the region of adverse pressure gradient, and thereafter, in the constant pressure region, were observed to decay very slowly. In contrast, the pressure rise in the flows studied by the present author was very abrupt, and it seems unlikely that the boundary layer approached equilibrium conditions during this rise. Furthermore, Bradshaw & Galea (1967) have observed that shear stresses are not immediately affected very much by steep pressure gradients, i.e. the response of the shear stress to pressure variation seems slow.

Thus we do not know how to expect a physically realistic adaptation to supersonic flow of Head's method to perform in the present test. The work of Bradshaw & Ferriss (1965) indicates that complete agreement cannot be expected but, for the reasons given above, it may well be an unreliable guide to the likely divergence.

Some test of the present two methods is needed against a type of flow for which, at low speeds, Head's method is known to give satisfactory results. The limited experimental data obtained in pressure gradients at supersonic speeds have been discounted because they involve appreciable pressure gradients $\partial p/\partial y$ across the boundary layer. We therefore resort to some of the numerous measurements obtained in equilibrium, constant pressure flows.

6. The shape parameter H_1 in compressible flows at constant pressure

6.1. *The predictions of methods based on a transformation*

The transformation described in §3.1 may be written in the general form

$$d\bar{y} = E\frac{\rho}{\rho_e}dy, \quad d\bar{x} = G\,dx, \quad \frac{\bar{u}}{\bar{u}_e} = \frac{u}{u_e}, \tag{6.1}$$

to which all transformations under which the stream function is invariant may be reduced.

Using a bar to denote all quantities in the transformed plane, we may write

$$\bar{\theta} = E\theta, \quad \bar{\Delta} = E\Delta. \tag{6.2}$$

With uniform external flow, E and G are constant in most transformations, including that of Mager, so that

$$\frac{d\theta}{dx} = \frac{G}{E}\frac{d\bar{\theta}}{d\bar{x}}, \quad \frac{d\Delta}{dx} = \frac{G}{E}\frac{d\bar{\Delta}}{d\bar{x}}. \tag{6.3}$$

Now in zero pressure gradient

$$\frac{d\bar{\theta}}{d\bar{x}} = \frac{\bar{c}_f}{2} \quad \text{and} \quad \frac{d\bar{\Delta}}{d\bar{x}} = F(\bar{H}_1) \tag{6.4}$$

so that

$$\frac{d\theta}{dx} = \frac{G}{E}\frac{\bar{c}_f}{2} = \frac{c_f}{2} \tag{6.5}$$

and

$$\frac{d\Delta}{dx} = \frac{G}{E}F(\bar{H}_1) = \frac{c_f}{\bar{c}_f}F(\bar{H}_1) = \frac{c_f}{\bar{c}_f}F(H_1), \tag{6.6}$$

since H_1 is invariant under the transformation.

Thus, according to the transformation, compressibility does not affect the relative rates of growth of mass flow and momentum thicknesses, and values of H_1 found in compressible flow should be the same as the values found in incompressible flows at equivalent Reynolds numbers. This is consistent with the hypothesis that the flow in the transformed plane is identical to some real compressible flow.

6.2. *The predictions of the direct calculation method*

Equations (6.5) and (6.6) obtained from the transformation may be contrasted with the zero pressure gradient versions of (4.5) and (4.4)

$$\frac{d\theta}{dx} = \frac{c_f}{2}, \tag{6.7}$$

$$\frac{d\Delta}{dx} = F((H_1)_k). \tag{6.8}$$

Although the two momentum equations are the same, the above entrainment rate $d\Delta/dx$ is higher than that predicted by the transformation by a factor† $\bar{c}_f/c_f$ if values of F are equal. We may note, moreover, that in flows with equal H_1, $F((H_1)_k)$ is greater than $F(H_1)$.

The immediate conclusion is that with increasing Mach number this analysis predicts an increase in the rate of growth of mass flow thickness relative to that of momentum thickness. As this will result in higher values of H_1, the value of $F((H_1)_k)$ might well be less than that in incompressible flow. Consequently the estimation of H_1 in compressible flow is not entirely straightforward.

An upper limit to H_1 may be obtained by assuming that the rate of entrainment is independent of Mach number. In this case $d\Delta/dx$ remains constant with increasing Mach number while $d\theta/dx$ decreases as $c_f/\bar{c}_f$. H_1 will therefore be roughly $\bar{c}_f/c_f$ times its value in incompressible flow.

A more realistic estimate of H_1 based on the direct calculation method would seem to be its equilibrium value for which, in zero pressure gradient, dH_1/dx is zero. In this case,

$$\frac{d\Delta}{dx} = F((H_1)_k) = H_1\frac{d\theta}{dx} = H_1\frac{c_f}{2}. \tag{6.9}$$

Using the velocity profile family of Thompson (1965) for incompressible flows, a value of H_1 of 8 is found to be typical of the incompressible flat plate boundary layer. The variation of H_1 with Reynolds number is small so that, using a bar now to denote quantities in the incompressible flow, it is a good approximation to write

$$\bar{F} = 8(\bar{c}_f/2). \tag{6.10}$$

From the last two equations we obtain

$$H_1 = \frac{8F((H_1)_k)}{\bar{F}}\frac{\bar{c}_f}{c_f}. \tag{6.11}$$

The results of the appendix are used to obtain $(H_1)_k$ given H_1, and $\bar{F}$ may be taken as $F(8)$.

Equation (6.11) thus provides an estimate of the equilibrium value of H_1. The earlier arguments, in which $d\Delta/dx$ was assumed independent of Mach number, provided an estimated upper limit for H_1,

$$H_1 = 8(\bar{c}_f/c_f). \tag{6.12}$$

† This is now the ratio of skin-friction coefficient in incompressible and compressible flows at 'corresponding' Reynolds numbers and, although the definition of corresponding presents difficulties, is a quantity which unmistakably increases with Mach number.

For the present purpose, the variation of $c_f/\bar{c}_f$ with Mach number has been obtained from the empirical relation of Spalding & Chi (1964), taking the flow to be adiabatic at a constant value of $R_x = 10^7$.

6.3. *Comparison with measurements in zero pressure gradient in supersonic flow*

The variation of H_1 with Mach number according to the above hypotheses is shown in figure 4, together with some experimental results. The line H_1 = constant predicted by the transformation is also shown.

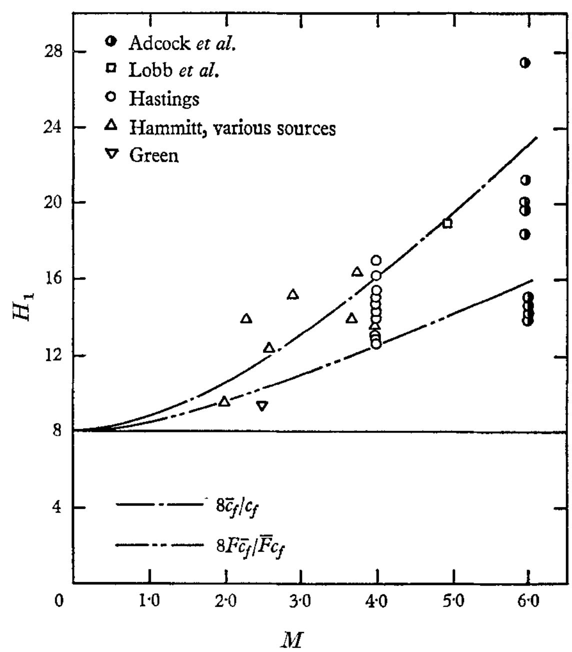


FIGURE 4. Variation of H_1 with Mach number in adiabatic flow at constant pressure.

The differences between the three hypotheses are large at higher Mach numbers and it is at once apparent that the experimental results cannot be reconciled with a transformation. In contrast the upper two lines, though their divergence indicates the crudity of their respective derivations, do lie within the spread of the experimental data and, more important, they broadly follow the trend with Mach number of these data.

Some of the results shown in figure 4 were obtained on wind tunnel walls and so are not strictly 'flat plate' measurements. Exceptions to this are the measurements by Hastings (1964) on a flat plate at $M = 4$, and those by Adcock, Peterson & McRee (1965) on the outside of a cylinder at $M = 6$. It is interesting that in both these cases the trend was for H_1 to start high and to decrease with downstream distance. That is, the trend in H_1 was broadly towards what has been suggested here to be an equilibrium value.

7. Further discussion of the compressibility transformation

7.1. *Coles's transformation and its variants*

For completeness, we shall finally consider the analysis in which Coles (1962), by discarding the previously accepted equality between stream functions, developed a transformation for which he claimed considerable generality provided the flow was bounded by a smooth wall. An important feature of his analysis was the 'Law of Corresponding Stations',

$$\bar{c}_f \bar{R}_\theta = \frac{\rho_e \mu_e}{\rho_w \mu_w} c_f R_\theta, \tag{7.1}$$

which provided a necessary (but not sufficient) equality between corresponding points in any two flows related by the transformation.

In considering supersonic flows in zero pressure gradient, Coles took the condition of constant pressure to be invariant against the transformation. His 'Law of Corresponding Stations' then became a *sufficient* identity between the two flows (in constant pressure incompressible flow sufficiently far from transition, the boundary layer is uniquely defined by a single non-dimensional quantity such as R_θ) and could be used to test the validity of the transformation.

For the present purpose, the right-hand side of (6.13) has been evaluated from typical experimental data obtained, well downstream of transition, by Hastings (1964) and by Adcock *et al.* (1965). In each case the 'corresponding station' in incompressible flow is at a moderate Reynolds number, well away from transition, and with a value of H_1 (evaluated from Thompson's (1965) profile family) slightly less than 8. Thus Coles's transformation shares with its simpler predecessors the failure to predict the observed (figure 4) trend of H_1 with M.

A similar failure has been noted by Baronti & Libby (1966), who found that velocity profiles measured in supersonic flow at constant pressure, when transformed, became qualitatively akin to profiles in an incompressible flow with favourable pressure gradient. They observed that Coles assumed *a priori*, on grounds of physical appeal, that the condition of constant pressure was invariant, but argued that Crocco's (1963) extension of Coles's work showed that this was not formally necessary. This led to their proposal that the counterpart of a supersonic constant pressure flow might be an incompressible flow with favourable pressure gradient.

The trend of data in figure 4 might lead us to adopt a similar position. Thus, if Coles's condition on constant pressure is abandoned, his 'Law of Corresponding Stations' is no longer a sufficient relation between the two flows. In fact, if the transformation is rigorously applied, an infinity of similar relations, involving Reynolds numbers based on characteristic lengths of the form

$$\int_0^\delta \frac{\rho}{\rho_e} f\left(\frac{u}{u_e}\right) dy$$

will be obtained. These may be restated as the invariance against the transformation of the ratio (e.g. H_1) of any two of these quantities.

We might therefore proceed by defining a corresponding station in incompressible flow by values of $\bar{c}_f \bar{R}_\theta$ and $\bar{H}_1$ ($= H_1$). From the high Mach number data in figure 4 it is clear that this corresponding incompressible flow will have experienced considerable favourable pressure gradient.

Crocco's work, however, suggests that this interpretation of the transformation is not permissible. His equation (2.25), which Baronti & Libby cite as formally allowing pressure gradients to intrude, is discussed by him at length in §3 of his paper. He concludes that '...it is almost exactly true that the condition of constant pressure is invariant against the transformation, as originally postulated by Coles'.

In fact, for adiabatic flows at constant pressure and without transpiration, this statement must be exactly true if Coles's transformation is to apply rigorously to the flow adjacent to the wall. Since friction at the wall is Newtonian, we may write Coles' equation (3.2)

$$\left(\frac{\partial \bar{\tau}}{\partial \bar{y}}\right)_w = \frac{\bar{\rho}^2 \bar{\mu} \sigma}{\rho_w^2 \mu_w \eta^3} \left(\left(\frac{\partial \tau}{\partial y}\right)_w - \frac{\tau_w}{\rho_w \mu_w} \left(\frac{\partial \rho \mu}{\partial y}\right)_w \right), \tag{7.2}$$

where σ and η are scaling factors of the transformation. For zero heat transfer,

$$\left(\frac{\partial T}{\partial y}\right)_w = \left(\frac{\partial \rho}{\partial y}\right)_w = \left(\frac{\partial \mu}{\partial y}\right)_w = 0$$

and we obtain

$$\left(\frac{\partial \bar{\tau}}{\partial \bar{y}}\right)_w = \frac{\bar{\rho}^2 \mu \sigma}{\rho_w^2 \mu_w \eta^3} \left(\frac{\partial \tau}{\partial \bar{y}}\right)_w. \tag{7.3}$$

At the wall, Coles' equations (1.2) and (1.4) reduce to

$$\left.\begin{aligned} 0 &= -\frac{dp}{dx} + \left(\frac{\partial \tau}{\partial y}\right)_w \\ \text{and} \quad 0 &= -\frac{d\bar{p}}{d\bar{x}} + \left(\frac{\partial \bar{\tau}}{\partial \bar{y}}\right)_w \end{aligned}\right\} \tag{7.4}$$

and we see that if the supersonic flow is at constant pressure all the terms in the above equations are zero.

This result applies to any transformation which involves the Howarth–Dorodnitsyn ordinate transformation, a velocity scaling factor which is independant of y, and Newtonian friction at the wall in both real and transformed flows. It is simply that for zero heat transfer $(\partial \rho/\partial y)_w$ and therefore $(d^2\bar{y}/dy^2)_w$ are zero. Thus if dp/dx is zero it follows that $d\bar{p}/d\bar{x}$ is zero because $(\partial \mu/\partial y)_w$ is zero in adiabatic high speed flow and therefore curvature of the velocity profile at the wall must be zero in both real and transformed planes.

At this point we may note a difficulty in Crocco's form of the transformation. For supersonic flow at constant pressure the three terms in his equation (2.25) must vanish. Consider however a virtually adiabatic flow in which the local heat transfer rate is vanishingly small but (because of a very small amount of upstream heat transfer) there is a small, finite stagnation enthalpy thickness. In this case the term in square brackets on the right-hand side of Crocco's equation (2.25) may become large, positive or negative. It seems physically unreasonable that the transformed version of this flow should be sensitive to changes in a negligibly small heat transfer rate—i.e. virtually identical flows, differing only in the detail of their heat transfer distributions, ought to remain virtually identical in the transformed plane. From Crocco's equation (2.25) it is clear that this result will be obtained only if both σ and η are independent of x.

But Coles† and Crocco, by their 'substructure' hypotheses, and Baronti & Libby by their 'sublayer' hypothesis, have proposed that σ should be a decreasing function of x. If the above argument is accepted we must disallow these hypotheses and, with them, the impressive collapse of skin-friction data achieved by Coles and the similarly impressive correlation of velocity profiles by Baronti & Libby.

The point at issue is not the ability of the transformation to provide a good approximate relation between certain properties of compressible and incompressible flows (which, with a suitable choice of its scaling factors, it certainly does). It is, rather, the claim by its originator that 'the transformation represents at every stage a genuine kinematic and dynamic correspondence between two real flows, both of which are capable of being observed experimentally'.

The emphasis here should be on the word 'real'. Coles points out that the turbulent shear stress is, in effect, defined by the (stated) fluid accelerations. Thus his transformation relates two hypothetical time-average flows which genuinely satisfy the equations of motion.‡ What remains open to question is whether, without fully understanding the structure of turbulent flows, we can say that, for an observed time-average compressible flow, the corresponding time-average flow in the transformed plane (although it satisfies the equations of motion) is *really* a possibility.

It is suggested here that a valid test of this question is the comparison between experimental values, in high and low speed adiabatic flows at constant pressure, of such supposed invariants as H_1.

7.2. *On the definition of boundary-layer thickness*

It might be objected that there are at least two reasons why H_1, since it involves the boundary-layer thickness δ, is a poor basis for such a test. First, the definition of δ is quite arbitrary. Secondly, its determination from a plot of experimental values of u/u_e against y is prone to error.

Baronti & Libby, however, discussing the failure of the transformation to describe the outer 'velocity defect' region, show that the actual definition of

† We should remember that Coles thought the energy equation had no significant part to play in the transformation for constant pressure flow. From this standpoint one might maintain that, since it follows from requiring a consistent transformation of the energy and momentum equations, Crocco's equation (2.25) is irrelevant and there is thus no argument for $d\sigma/dx$ being zero.

‡ If, as mooted by Baronti & Libby, an adiabatic high speed flow at constant pressure were transformed to an accelerating low speed flow, physical realism would have to be compromised at the wall. The mapping to low speed would yield a velocity profile with zero curvature at the wall but a non-zero value of $(\partial\bar{\tau}/\partial\bar{y})_w$, the equations of motion would be satisfied but the shear-stress and velocity fields adjacent to the wall would be physically incompatible.

δ is qualitatively unimportant, 'the discrepancy in the correlation of the outer layer cannot be accounted for on this basis'.

Figure 5 shows velocity profiles measured in incompressible flow and at Mach 6 by, respectively, Smith & Walker (1958) and Adcock *et al.* (1965). The flows are adiabatic, at constant pressure, and for both profiles the quantity $(\rho_e\mu_e/\rho_w\mu_w)\,c_f R_\theta$ is 13.

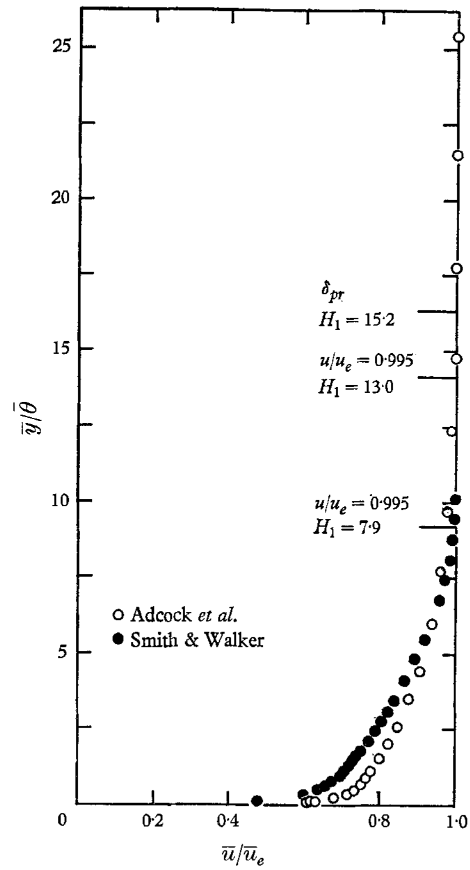


FIGURE 5. Comparison between two velocity profiles which satisfy Coles's 'Law of Corresponding Stations', one measured in incompressible flow and the other at Mach 6.

The axes are $$\bar{u}/\bar{u}_e = u/u_e$$
and $$\frac{\bar{y}}{\theta} = \int_0^y \left(\frac{\rho}{\rho_e}\right) dy \Big/ \theta.$$

These two equalities hold good for every transformation discussed in this paper, and at corresponding stations the functions

$$\frac{\bar{u}}{\bar{u}_e}\left(\frac{\bar{y}}{\theta}\right) \quad \text{and} \quad \frac{u}{u_e}\left(\int_0^y \left(\frac{\rho}{\rho_e}\right) dy \Big/ \theta\right)$$

should therefore be identical. Clearly this is not so in figure 5.

The values of $\bar{y}$ at $\bar{u}/\bar{u}_e$ and $u/u_e = 0{\cdot}995$ are shown, and also δ_{pr}, the value obtained by Adcock *et al.* (and which is used in figure 4) by extrapolating the linear portion of the Pitot pressure distribution.

We see that $\bar{\delta}/\bar{\theta}$ for the compressible boundary layer is almost twice that of the corresponding incompressible layer, and that the 'tail' beyond $\bar{y} = \bar{\delta}$ is very much greater for the compressible layer. Also, the compressible profile is much fuller near the wall (a perhaps surprising result in view of Baronti & Libby's successful correlation of the wall region). In fact, though the fullness of incompressible profiles increases with Reynolds number, and that shown in figure 5 is at one of the lowest Reynolds numbers measured by Smith & Walker, none of these authors' measured profiles, even at their highest Reynolds number, is fuller than the Mach 6 profile in figure 5.

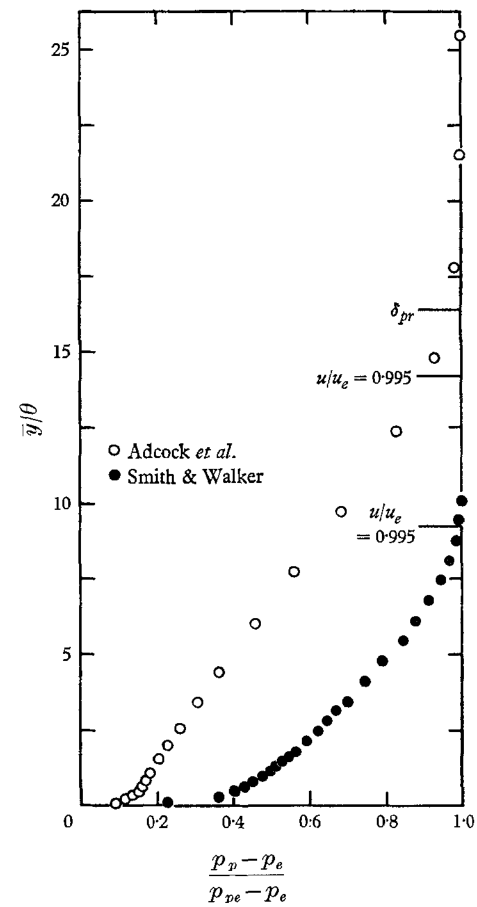


FIGURE 6. Pitot profiles measured in incompressible flow and at Mach 6, showing the much greater thickness of the high speed boundary layer.

Because, at high Mach number, the velocity defect is so small over the outer part of the layer, there is a danger that it may be masked by small errors in temperature measurement. This does not mean, however, that there is a large uncertainty as to the thickness of the layer. Figure 6 shows the Pitot pressure distributions for the two profiles shown in figure 5. There can be little doubt that the region into which vorticity is transported is about twice as thick at $M = 6$ as the transformation would lead us to expect.

8. Concluding remarks

In any attempt to predict the time-average properties of a (hydrodynamic) turbulent boundary layer in low speed flow, by far the greatest uncertainty lies in quantifying the influence which the turbulent motion has on the time-average flow. This paper has demonstrated, by considering two hypotheses, neither of which is obviously unreasonable, that in compressible flow this uncertainty increases appreciably with increasing Mach number.

Considering flows at constant pressure, we find it implicit in compressibility transformations that entrainment rate (a crude measure of turbulence effects) and skin-friction coefficient decrease proportionately with increasing Mach number. On the other hand, Head's original suggestions lead to a model of the flow in which the fractional reduction in entrainment is appreciably less than that in skin-friction coefficient.

The available data from experiments in constant pressure flows reveal a variation of the shape parameter H_1 with Mach number which endorses the second of these views. Moreover, there is broad quantitative agreement between theory and experiment on the extent of this variation. In contrast, compressibility transformations imply no variation. It is shown that this feature is common to all transformations, and results in a gross underestimate of boundary-layer thickness at high Mach numbers.

Further support for the second treatment is found in comparisons with the observed boundary-layer development downstream of an abrupt pressure rise. Accordingly, this direct extension of Head's method is put forward as an interim calculation procedure for supersonic flows.

This work was performed in the Aeronautics Sub-Department of the Cambridge University Engineering Laboratory. It was supervised by Mr E. P. Sutton and Dr L. C. Squire, and profitably discussed with Dr M. R. Head who is also acknowledged in the text. The author is extremely grateful for all the advice and encouragement he received, and wishes also to thank Mr R. C. Hastings for access to some unpublished data and the then D.S.I.R. for financial support.

Appendix. Relations between 'transformed' and 'kinematic' shape parameters for a compressible boundary layer

We are concerned with relations between integral parameters of the form

$$I_k = \int_0^\delta \phi\left(\frac{u}{u_e}\right) dy, \tag{A 1}$$

which will here be referred to as 'kinematic' parameters, and corresponding 'transformed' parameters of the form

$$\bar{I} = \int_0^{\bar{\delta}} \phi\left(\frac{u}{u_e}\right) d\bar{y} = \int_0^\delta \frac{\rho}{\rho_e} \phi\left(\frac{u}{u_e}\right) dy, \tag{A 2}$$

where we have made the substitutions

$$d\bar{y} = \frac{\rho}{\rho_e} dy, \quad \bar{\delta} = \int_0^\delta \frac{\rho}{\rho_e} dy.$$

The author has treated this problem in general terms elsewhere (Green 1966) and will present only two particular results here. These results are for boundary layers in which the relation between velocity and temperature is the parabola

$$\frac{T}{T_e} = 1 + r\left(\frac{\gamma-1}{2}\right)M_e^2 - r\left(\frac{\gamma-1}{2}\right)M_e^2\left(\frac{u}{u_e}\right)^2, \tag{A 3}$$

i.e. for flows with an adiabatic wall and recovery factor r or, writing $r = 1$, for iso-energetic flows. They apply to flows in which the velocity profile in the transformed plane fits the power law relation

$$u/u_e = (\bar{y}/\bar{\delta})^{1/n}. \tag{A 4}$$

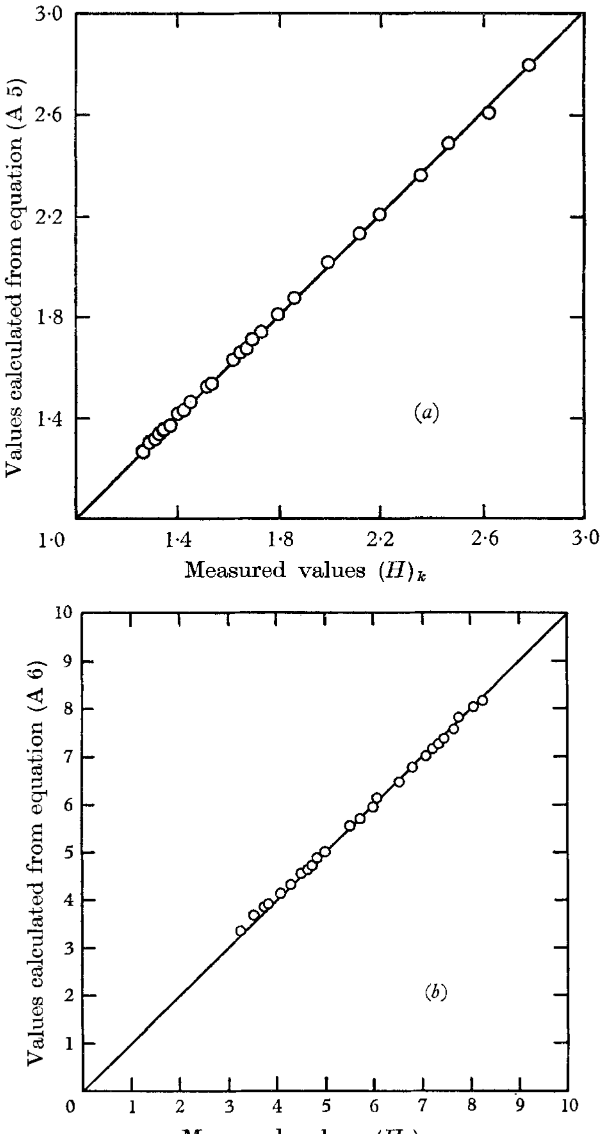


FIGURE 7. Relations between transformed and kinematic shape parameters for power-law velocity profiles and iso-energetic flow, comparison with experiment.

For $\gamma = 1{\cdot}4$, the following two expressions are obtained:

$$H_k = \bar{H}\left[1+\frac{\dfrac{rM_e^2}{5}\dfrac{(\bar{H}+1)(\bar{H}-1)^2}{\bar{H}(3\bar{H}-1)(2\bar{H}-1)}}{1+\dfrac{rM_e^2}{5}\left\{1-\dfrac{\bar{H}(\bar{H}+1)}{(3\bar{H}-1)(2\bar{H}-1)}\right\}}\right], \tag{A 5}$$

which is used in the calculation method of §4, and

$$\frac{H_1}{(H_1)_k} = 1+\left(\frac{H_1-1}{H_1+2}\right)\frac{\dfrac{rM_e^2}{5}}{\left(\dfrac{rM_e^2}{5}+\left(\dfrac{H_1+1}{2}\right)\right)}. \tag{A 6}$$

In figure 7 these relations are tested against the author's measurements in the region downstream of a shock reflexion.

In figure 7*a* measured values of $(H)_k$ are plotted against values deduced from (A 5) using measured values of $\bar{H}$ and assuming a recovery factor of 1. In figure 7*b* measured values of $(H_1)_k$ are plotted against values deduced from (A 6). In both cases agreement is satisfactory and adequately justifies the use made of this analysis in the calculation method of §4.

REFERENCES

Adcock, J. B., Peterson, J. B. & McRee, D. I. 1965 *NASA TN* D-2907.

Baronti, P. O. & Libby, P. A. 1966 *AIAA J.* **4**, 193.

Bradshaw, P. & Ferriss, D. H. 1965 *ARC* 26,758.

Bradshaw, P. & Ferriss, D. H. 1966 *ARC* 28541.

Bradshaw, P. & Galea, P. V. 1967 *J. Fluid Mech.* **27**, 111.

Coles, D. E. 1956 *J. Fluid Mech.* **1**, 191.

Coles, D. E. 1962 *USAF Project Rand Rept.* R-403-PR (also *Physics of Fluids*, **7**, 1403).

Crocco, L. 1963 *AIAA J.* **1**, 2723.

Culick, F. E. C. & Hill, J. A. F. 1958 *J. Aero. Sci.* **25**, 259.

Eckert, E. R. G. 1955 *J. Aero. Sci.* **22**, 585.

Escudier, M. P. & Nicoll, W. B. 1966 *J. Fluid Mech.* **25**, 337.

Green, J. E. 1966 Ph.D. Thesis, Cambridge University.

Hammitt, A. G. 1958 *J. Aero. Sci.* **25**, 345.

Hastings, R. C. 1964 R.A.E. (unpublished).

Head, M. R. 1960 *ARC R & M* 3152.

Lobb, R. K., Winkler, E. M. & Persh, J. 1955 *J. Aero. Sci.* **22**, 1.

Ludwieg, H. & Tillmann, W. 1949 *Ing. Archiv.* **17**, 288 (also *NACA TM* 1285 (1950)).

Mager, A. 1958 *J. Aero. Sci.* **25**, 305.

Mager, A. 1962 *J. Aerospace Sci.* **29**, 752.

Maise, G. & McDonald, H. 1967 *AIAA Paper* 67–199.

Morkovin, M. V. 1962 *Mecanique de la Turbulence*, Colloques Intern. du C.N.R.S. no. 108, 367. (Also available as *The Mechanics of Turbulence*. New York: Gordon and Breach.)

Patel, V. C. 1965 Ph.D. Thesis, Cambridge University.

Smith, D. W. & Walker, J. H. 1958 *NACA TN* 4231.

So, R. M. C. 1965 McGill Univ. Tech Note 65–6.

Spalding, D. B. & Chi, S. W. 1964 *J. Fluid Mech.* **18**, 117.
Spence, D. A. 1961 *ARC R & M* 3191.
Squire, W. 1962 *J. Aerospace Sci.* **29**, 237.
Standen, N. M. 1964 *AIAA Paper* no. 64–584.
Stewartson, K. 1949 *Proc. Roy. Soc.* A **200**, 84.
Thompson, B. G. J. 1964 *ARC R & M* 3447.
Thompson, B. G. J. 1965 *ARC R & M* 3463.